Determinism in synthesized chaotic waveforms

Ned J. Corron, Jonathan N. Blakely, Scott T. Hayes, and Shawn D. Pethel
U.S. Army RDECOM, AMSRD-AMR-WS-ST, Redstone Arsenal, Alabama 35898, USA
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The output of a linear filter driven by a randomly polarized square wave, when viewed backward in time, is shown to exhibit determinism at all times when embedded in a three-dimensional state space. Combined with previous results establishing exponential divergence equivalent to a positive Lyapunov exponent, this result rigorously shows that such reverse-time synthesized waveforms appear equally to have been produced by a deterministic chaotic system.

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Recently, a surprising new mechanism for generating chaotic waveforms was reported [1,2]. Rather than arising from the evolution of nonlinear dynamical systems, it was shown that chaotic waveforms can result from the convolution of a specially constructed basis pulse with an infinite train of randomly polarized delta functions. In this context there is no chaotic attractor, yet the output waveform exhibits all the salient properties of a waveform generated by a chaotic dynamical system, including a positive Lyapunov exponent and determinism when embedded in a suitable state space [3]. Importantly, it was found that this mechanism can occur in very simple physical systems [4]. In particular, it was shown that a linear, second-order filter excited by a randomly polarized square wave can generate a waveform that, when viewed backward in time, is chaotic. This output waveform was called reverse-time chaos. Furthermore it was found that the same filter can produce a number of different chaotic topologies, including a folded band structure [5] and multilobed sets [6]. The discovery that chaos can arise outside the context of deterministic nonlinear systems raises the intriguing possibility that chaos may play an unexpected role in physical theories not based on nonlinear dynamic models.

In the initial description of reverse-time chaos, it was shown that the waveform exhibits a shift map when sampled at the drive transitions. Although this demonstrated a positive Lyapunov exponent and determinism in the waveform samples, this result did not address what happened between sample times. In particular, it remained unclear whether determinism is exhibited at all times in the continuous-time waveform. In this Brief Report we present a more rigorous proof of chaos by explicitly showing determinism at all times when the waveform is embedded in $\mathfrak{R}^3$.

A previous paper considered the response of the driven linear filter

$$\ddot{x} + 2\beta\dot{x} + (\omega^2 + \beta^2)x = s(t), \tag{1}$$

where $\beta = \ln 2$, $\omega = 2k\pi$, and $s(t)$ is a randomly polarized square wave [4]. Specifically, the filter input is

$$s(t) = s_n, \quad n \le t < n+1, \tag{2}$$

where the polarity of each unit-length square pulse is represented by the random sequence $\{s_n\}$, with each symbol $s_n \in \{-1,+1\}$. The system (1), (2) yields the explicit solution

$$x(t) = \frac{s_{[t]} + 2^{[t]-t}\left(\cos\omega t + \dfrac{\beta}{\omega}\sin\omega t\right)\left(-s_{[t]} + \sum_{i=1}^{\infty} s_{[t]-i}2^{-i}\right)}{\omega^2 + \beta^2}, \tag{3}$$

where $[t]$ represents the largest integer less than or equal to t. In the previous work, the frequency was allowed to take values from a discrete spectrum with $k \in \{1,2,\ldots\}$, but here we primarily consider just the fundamental $k=1$. An important conclusion of the previous work is that the waveform (3) exhibits reverse-time chaos; that is, when viewed backward in time, the waveform appears equally to have been generated by a chaotic dynamical system. To support this conclusion, it was shown that the reverse-time waveform sampled at integer times $t=n$ satisfies a chaotic shift map, thereby demonstrating determinism and a positive Lyapunov exponent ($\lambda = \ln 2$) in the waveform samples.

The previous analysis left uncertain whether the reverse-time waveform is deterministic at times other than sample times. In this report, we show that the waveform exhibits determinism at all times when embedded in $\mathfrak{R}^3$. Specifically, this result means that, for the waveform viewed in reverse time, specifying an initial condition—namely, the state x and its first two derivatives $\dot{x}$ and $\ddot{x}$—uniquely determines the future trajectory of the waveform. Equivalently, when viewing the filter response (3) in forward time, the state and two derivatives specified at a final time t_0 uniquely describe the complete history of the filter state.

To begin, we suppose that $x(t_0)$, $\dot{x}(t_0)$, and $\ddot{x}(t_0)$ are given at an unspecified time t_0 as shown in Fig. 1. Without loss of generality, we assume that $0 \le t_0 < 1$; in effect, we identify t_0

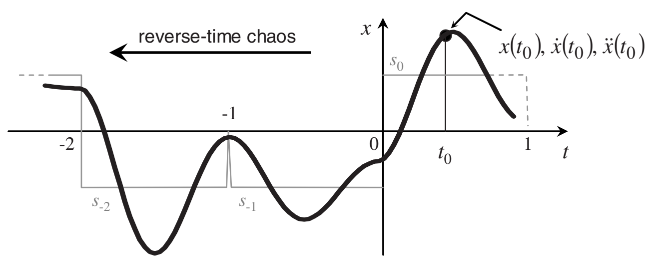

FIG. 1. Initial conditions specified at $t=t_0$ for a reverse-time chaotic waveform $x(t)$.

as the unknown phase of the initial conditions relative to the drive sequence in Eq. (2). To prove determinism in the reverse-time waveform, it is sufficient to show that the given conditions uniquely specify the state $x(t)$ via Eq. (3) for all times $t<t_0$.

Evaluating equation (1) at $t=t_0$, we find

$$s_0 = \ddot{x}(t_0) + 2\beta\dot{x}(t_0) + (\omega^2+\beta^2)x(t_0) \tag{4}$$

implying the current symbol s_0 is uniquely and immediately determined by the given conditions. A derivative of the solution (3) yields

$$\dot{x}(t) = \frac{-2^{[t]-t}}{\omega}\left(-s_{[t]} + \sum_{i=1}^{\infty} s_{[t]-i}2^{-i}\right)\sin \omega t \tag{5}$$

which is valid for all t. Although there is an apparent discontinuity in Eq. (3) at $t=0$, we know by construction that the solution (3) and its first derivative (5) are continuous everywhere [4]. Evaluating Eqs. (3) and (5) at $t=t_0$ yields

$$(\omega^2+\beta^2)x(t_0) = s_0 + 2^{-t_0}(r-s_0)\left(\cos \omega t_0 + \frac{\beta}{\omega}\sin \omega t_0\right),$$

$$\dot{x}(t_0) = -\frac{2^{-t_0}}{\omega}(r-s_0)\sin \omega t_0, \tag{6}$$

where we used $[t_0]=0$. In Eq. (6), we have defined

$$r = \sum_{i=1}^{\infty} s_{-i}2^{-i}. \tag{7}$$

Since each symbol $s_{-i}=\pm 1$ for $i=1,\ldots,\infty$, the series (7) is necessarily bounded as $-1\leq r\leq 1$. Moreover, any value of r in this range maps to a unique sequence of symbols s_{-i} via the binary representation (7). As a result, determining r implies all prior symbols are also determined.

Proving determinism is now reduced to showing that t_0 and r are uniquely determined by the given conditions. To this end, we define the intermediate quantities

$$A = 2^{-t_0}(r-s_0)\cos \omega t_0,$$

$$B = 2^{-t_0}(r-s_0)\sin \omega t_0. \tag{8}$$

Using Eq. (6), we see that

$$A = (\omega^2+\beta^2)x(t_0) + \beta\dot{x}(t_0) - s_0,$$

$$B = -\omega\dot{x}(t_0), \tag{9}$$

where everything on the right-hand side is known. Thus, the intermediate quantities A and B are uniquely determined by the given conditions. From Eq. (8), we find

$$r - s_0 = \pm 2^{t_0}\sqrt{A^2+B^2}. \tag{10}$$

The ambiguity in the sign in Eq. (10) is resolved by recalling that $-1\leq r\leq 1$ and $s_0=\pm 1$. If $s_0=+1$, then $-2\leq r-s_0\leq 0$. Similarly, if $s_0=-1$, then $0\leq r-s_0\leq 2$. Thus, the sign of $r-s_0$ is determined entirely by s_0,

$$\text{sgn}(r-s_0) = -s_0 \tag{11}$$

and

$$r = s_0(1-2^{t_0}\sqrt{A^2+B^2}). \tag{12}$$

Thus, if t_0 is determined then r is fixed as well.

To determine t_0, we use Eqs. (8) and (12) to get

$$\cos(\omega t_0) = \frac{-s_0A}{\sqrt{A^2+B^2}},$$

$$\sin(\omega t_0) = \frac{-s_0B}{\sqrt{A^2+B^2}}, \tag{13}$$

where by assumption we have $0\leq t_0<1$. Formally, we may write

$$t_0 = \frac{1}{\omega}\tan^{-1}\left(\frac{B}{A}\right) \tag{14}$$

but the solution is ambiguous with period $\frac{1}{2}$ for $k=1$. This phase ambiguity is resolved by considering the sign of the cosine and sine in Eq. (13). Here it is sufficient to note that the two expressions in Eq. (13) admit a unique solution in the range $0\leq t_0<1$. Thus t_0 is uniquely determined, and the prior symbols are likewise determined via Eqs. (7) and (12).

Thus we have shown that the given conditions $x(t_0)$, $\dot{x}(t_0)$, and $\ddot{x}(t_0)$ uniquely specify the unknown phase t_0, the current symbol s_0, as well as all prior symbols $s_{-1}, s_{-2},\ldots$, which is everything necessary to evaluate Eq. (3) for any $t<t_0$. Consequently, it follows that the waveform (3) exhibits reverse-time determinism at all times when embedded in $\mathfrak{R}^3$. Combined with previous results establishing exponential divergence and a positive Lyapunov exponent [4], this result rigorously shows that the linearly synthesized waveform (3), when viewed backward in time, is chaotic.

We note that $\mathfrak{R}^3$ is a sufficient embedding dimension for the chaotic waveform. However, the present analysis does not establish a minimum embedding dimension for the synthesized waveform, and it may be that a three-dimensional embedding is not strictly necessary. We also note that the ambiguity in Eq. (14) is resolved only for $k=1$, and this proof of determinism is incomplete for $k>1$. In such cases, a different embedding or possibly a larger embedding dimension may be required to prove determinism at all times.

It is significant to recognize that the synthesized waveform (3) exhibits determinism in only one direction, namely, backward in time. In fact, the reverse-time chaotic waveform is not invertible; in the preceding analysis, the sequence of future symbols $s_1,s_2,\ldots$, is left unspecified by the given conditions. In this respect the reverse-time chaos in Eq. (3) differs qualitatively from chaos generated by continuous low-dimensional flows such as the Lorenz or Rössler systems, which exhibit determinism in both the past and future [7]. Instead, the synthesized waveform exhibits the characteristics of a semiflow, which is typical in delay-dynamical systems and discontinuous vector fields [8,9]. Here, one-sided determinism is a necessary consequence of causality in the physically realizable filter of Eq. (1). However, it has been shown elsewhere that chaotic waveforms can also be synthesized using an acausal basis function with exponential decay in both future and past directions [1]. A chaotic waveform synthesized with a two-sided pulse can be invertible, and a rigorous proof of two-sided determinism in a class of these synthesized chaotic waveforms is the subject of future work [10].

[1] S. T. Hayes, J. Phys.: Conf. Ser. **23**, 215 (2005).
[2] Y. Hirata and K. Judd, Chaos **15**, 033102 (2005).
[3] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, New York, 1993).
[4] N. J. Corron, S. T. Hayes, S. D. Pethel, and J. N. Blakely, Phys. Rev. Lett. **97**, 024101 (2006).
[5] N. J. Corron, S. T. Hayes, S. D. Pethel, and J. N. Blakely, Phys. Rev. E **75**, 045201(R) (2007).
[6] N. J. Corron, S. T. Hayes, S. D. Pethel, and J. N. Blakely, *IEEE International Symposium on Circuits and Systems 2007 (ISCAS 2007)* (IEEE, Piscataway, NJ, 2007), pp. 205–208.
[7] J. M. T. Thompson and H. B. Stewart, *Nonlinear Dynamics and Chaos* (Wiley, Chichester, UK, 1986).
[8] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, *Delay Equations* (Springer-Verlag, New York, 1995).
[9] E. Sander, E. Barreto, S. J. Schiff, and P. So, Discrete Contin. Dyn. Syst. **2005**, 768 (2005).
[10] S. T. Hayes (unpublished).